

Loop super-Virasoro Lie conformal superalgebra

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Abstract: The loop super-Virasoro conformal superalgebra \mathfrak{cls} associated with the loop super-Virasoro algebra is constructed in the present paper. The conformal superderivation algebra of \mathfrak{cls} is completely determined, which is shown to consist of inner superderivations. And nontrivial free and free \mathbb{Z} -graded \mathfrak{cls} -modules of rank two are classified. We also give a classification of irreducible free \mathfrak{cls} -modules of rank two and all irreducible submodules of each free \mathbb{Z} -graded \mathfrak{cls} -module of rank two.

Key words: loop super-Virasoro algebra, Lie conformal superalgebras, conformal superderivations, conformal modules.

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1 Introduction

Lie conformal (super)algebras, originally introduced by Kac in [12, 13], encode an axiomatic description for the singular part of the operator product expansion of chiral fields in two-dimensional conformal field theory. They are very closely related to vertex algebras (cf. [1, 15]) by the same way as Lie algebras correspond to their universal enveloping algebras. On the other hand, the theory of Lie conformal (super)algebras give us powerful tools for the study of infinite-dimensional Lie (super)algebras and associative algebras satisfying the locality property described in [11]. The conformal (super)algebras have drawn much attention in branches of physics and mathematics since the introduction. The structure theory, representation theory and cohomology theory of finite (i.e., finitely generated as $\mathbb{C}[\partial]$ -modules) Lie conformal algebras have been well developed (cf. [2, 5, 6, 8]), and finite simple Lie conformal superalgebras were classified in [9] and their representation theories were developed in [3, 4, 14].

The object investigated in this paper is a Lie conformal superalgebra closed related to the loop super-Virasoro algebra \mathfrak{sl} whose structures were studied in [7]. It is defined as a infinite-dimensional Lie superalgebra with a basis $\{L_{\alpha,i}, G_{\mu,j} \mid \alpha, i, j \in \mathbb{Z}, \mu \in \frac{1}{2} + \mathbb{Z}\}$ satisfying the following commutation relations:

$$[L_{\alpha,i}, L_{\beta,j}] = (\alpha - \beta)L_{\alpha+\beta,i+j}, \quad [L_{\alpha,i}, G_{\mu,j}] = \left(\frac{\alpha}{2} - \mu\right)G_{\alpha+\mu,i+j}, \quad [G_{\mu,i}, G_{\nu,j}] = 2L_{\mu+\nu,i+j}, \quad (1.1)$$

the even and odd parts of which are $\mathfrak{sl}^{\bar{0}} = \text{span}\{L_{\alpha,i} \mid \alpha, i \in \mathbb{Z}\}$ and $\mathfrak{sl}^{\bar{1}} = \text{span}\{G_{\mu,j} \mid j \in \mathbb{Z}, \mu \in \frac{1}{2} + \mathbb{Z}\}$, respectively. Clearly, $\mathfrak{sl}^{\bar{0}}$ is just the loop Virasoro algebra (cf. [10]) and $\mathfrak{sl}^{\bar{1}}$ is

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its module. Hence, the loop super-Virasoro algebra can be seen as a super extension of the loop Virasoro algebra.

Motivated by the idea from [13] we associate a Lie conformal superalgebra with the loop super-Virasoro algebra. It is called the *loop super-Virasoro conformal superalgebra*, denoted by \mathbf{cls} , which is a $\mathbb{C}[\partial]$ -module $\mathbf{cls} = (\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]L_i) \oplus (\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]G_i)$ with a $\mathbb{C}[\partial]$ -basis $\{L_i, G_i \mid i \in \mathbb{Z}\}$ satisfying the following λ -brackets:

$$[L_i \lambda L_j] = (\partial + 2\lambda)L_{i+j}, \quad [L_i \lambda G_j] = (\partial + \frac{3}{2}\lambda)G_{i+j}, \quad (1.2)$$

$$[G_i \lambda L_j] = (\frac{1}{2}\partial + \frac{3}{2}\lambda)G_{i+j}, \quad [G_i \lambda G_j] = 2L_{i+j}, \quad \forall i, j \in \mathbb{Z}. \quad (1.3)$$

Note that this is an infinite simple Lie conformal superalgebra, containing the loop Virasoro Lie conformal algebra $\mathbf{clv} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]L_i$ (see [16]) as its subalgebra. As pointed out previously, the theory of finite simple Lie conformal (super)algebras were well developed, but so far there is no systematic theory for the infinite case. So it is interesting and necessary to develop the theory for infinite simple Lie conformal superalgebras. This is one of our motivations for studying the loop super-Virasoro conformal superalgebra. We shall study the superderivation algebra of \mathbf{cls} and free (\mathbb{Z} -graded) \mathbf{cls} -modules of rank ≤ 2 . One interesting aspect is that free (\mathbb{Z} -graded) \mathbf{cls} -modules of rank 1 are trivial, which is totally different from the loop Virasoro Lie conformal algebra case (all $V_{a,b}$ and $V_{A,b}$ are its nontrivial conformal modules of rank one); and the other interesting aspect is that the even or odd part of a \mathbb{Z} -grade free \mathbf{cls} -module of rank two has the form $V_{A,b}$ if and only if the other part is $A_{\frac{1}{2},b}$. We remark that an important strategy frequently used in the present paper is to pass modules over \mathbf{cls} to modules over \mathbf{clv} .

This paper is organized as follows. In Sect. 2, we collect some facts and notions related to Lie conformal superalgebras. In Sect. 3, we determinate conformal superderivations of \mathbf{cls} . The Sect. 4 is devoted to giving the classification of all nontrivial free \mathbf{cls} -modules of rank less than two. We also determine the irreducibility of these modules and therefore classify all inequivalent irreducible free \mathbf{cls} -modules of rank two. And free \mathbb{Z} -graded \mathbf{cls} -modules of rank less than or equal to two are classified in Sect. 5. Moreover, all irreducible submodules of free \mathbb{Z} -graded \mathbf{cls} -modules of rank two are completely determined.

Throughout this paper, all vector spaces are assumed to be over the complex field \mathbb{C} and all linear maps are \mathbb{C} -linear. The main results of this paper are summarized in Theorems 3.1, 4.2, 4.4 and 5.2.

2 Preliminaries

In this section, we recall some notions related to Lie conformal superalgebras and conformal modules from [8, 12, 13].

We say that a vector space U is \mathbb{Z}_2 -graded if $U = U^{\bar{0}} \oplus U^{\bar{1}}$, and $x \in U^{\bar{i}}$ is called \mathbb{Z}_2 -homogenous and write $|x| = i$. For any two \mathbb{Z}_2 -graded vector spaces U and V , a linear map $f : U \rightarrow V$ is called homogenous of degree $\bar{i} \in \mathbb{Z}_2$ if $f(U^{\bar{j}}) \subseteq V^{\bar{i}+\bar{j}}$ for all $\bar{j} \in \mathbb{Z}_2$.

Definition 2.1. A Lie conformal superalgebra is a \mathbb{Z}_2 -graded $\mathbb{C}[\partial]$ -module \mathcal{A} endowed with a linear map $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{A}$, $a \otimes b \mapsto [a_\lambda b]$, called the λ -bracket, and satisfying the following axioms ($a, b, c \in \mathcal{A}$) :

$$[(\partial a)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda b] = -(-1)^{|a||b|}[b_{-\lambda-\partial}a], \quad (2.1)$$

$$[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + (-1)^{|a||b|}[b_\mu[a_\lambda c]]. \quad (2.2)$$

It follows from the axioms in (2.1) that

$$[(f(\partial)a)_\lambda b] = f(-\lambda)[a_\lambda b] \text{ and } [a_\lambda(f(\partial)b)] = f(\partial + \lambda)[a_\lambda b], \quad \forall f(\partial) \in \mathbb{C}[\partial].$$

Definition 2.2. A conformal module over a Lie conformal superalgebra \mathcal{A} or an \mathcal{A} -module is a \mathbb{Z}_2 -graded $\mathbb{C}[\partial]$ -module V endowed with a λ -action $\mathcal{A} \otimes V \rightarrow \mathbb{C}[\lambda] \otimes V$, $a \otimes v \mapsto a_\lambda v$, and satisfying the following axioms ($a, b \in \mathcal{A}, v \in V$) :

$$(\partial a)_\lambda v = -\lambda a_\lambda v, \quad a_\lambda(\partial v) = (\partial + \lambda)a_\lambda v,$$

$$[a_\lambda b]_{\lambda+\mu}v = a_\lambda(b_\mu v) - (-1)^{|a||b|}b_\mu(a_\lambda v).$$

The rank of an \mathcal{A} -module V is defined to be the rank of V as $\mathbb{C}[\partial]$ -module.

Definition 2.3. Let \mathcal{A} be a Lie conformal superalgebra.

- (1) \mathcal{A} is called \mathbb{Z} -graded if $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i$, each \mathcal{A}^i is a $\mathbb{C}[\partial]$ -submodule and $[\mathcal{A}^i {}_\lambda \mathcal{A}^j] \subseteq \mathcal{A}^{i+j}[\lambda]$ for any $i, j \in \mathbb{Z}$.
- (2) A conformal module V over \mathcal{A} is \mathbb{Z} -graded if $V = \bigoplus_{i \in \mathbb{Z}} V_i$, each V_i is a $\mathbb{C}[\partial]$ -submodule and $(\mathcal{A}^i)_\lambda V_j \subseteq V_{i+j}[\lambda]$ for any $i, j \in \mathbb{Z}$. If each V_i is a free $\mathbb{C}[\partial]$ -module of rank n , then V is called a free \mathbb{Z} -graded \mathcal{A} -module of rank n .

Note that the loop super-Virasoro conformal superalgebra $\mathbf{cls} = (\mathbf{cls})^{\bar{0}} \oplus (\mathbf{cls})^{\bar{1}}$ is \mathbb{Z}_2 -graded with $(\mathbf{cls})^{\bar{0}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]L_i$ and $(\mathbf{cls})^{\bar{1}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]G_i$ such that $[(\mathbf{cls})^\alpha {}_\lambda (\mathbf{cls})^\beta] \subseteq (\mathbf{cls})^{\alpha+\beta}[\lambda]$ for any $\alpha, \beta \in \mathbb{Z}_2$, and on the other hand $\mathbf{cls} = \bigoplus_{i \in \mathbb{Z}} (\mathbf{cls})_i$ is also \mathbb{Z} -graded with $(\mathbf{cls})_i = \mathbb{C}[\partial]L_i \oplus \mathbb{C}[\partial]G_i$ for each i in the sense of Definition 2.3.

3 Conformal superderivations

A homogenous linear map $D_\lambda : \mathcal{A} \rightarrow \mathcal{A}[\lambda]$ of degree $\bar{i} \in \mathbb{Z}_2$ is called a *homogeneous conformal superderivation of degree \bar{i}* if the following conditions hold:

$$D_\lambda(\partial a) = (\partial + \lambda)D_\lambda a, \quad D_\lambda[a_\mu b] = [[D_\lambda a]_{\lambda+\mu}b] + (-1)^{i|a|}[a_\mu[D_\lambda b]], \quad \forall a, b \in \mathcal{A}.$$

And we write D instead of D_λ for simplicity. Denote the set of all conformal superderivations of degree $\alpha \in \mathbb{Z}_2$ by $\text{CDer}^\alpha(\mathcal{A})$. Then we call $\text{CDer}(\mathcal{A}) = \text{CDer}^{\bar{0}}(\mathcal{A}) \oplus \text{CDer}^{\bar{1}}(\mathcal{A})$ the conformal superderivation algebra of \mathcal{A} and each element of $\text{CDer}(\mathcal{A})$ a superderivation of \mathcal{A} . For any $a \in \mathcal{A}$, one can see easily that the linear map $(\text{ad}_a)_\lambda : \mathcal{A} \rightarrow \mathcal{A}[\lambda]$ given by $(\text{ad}_a)_\lambda b = [a_\lambda b]$ for $b \in \mathcal{A}$ is a conformal superderivation, which is called an *inner conformal superderivation* of \mathcal{A} . Denote the set of all inner conformal superderivations of \mathcal{A} by $\text{CInn}(\mathcal{A})$.

Now we are ready to give the main result of this section, which establishes the equality between the two sets $\text{CDer}(\mathfrak{cls})$ and $\text{CInn}(\mathfrak{cls})$.

Theorem 3.1. *Every conformal superderivation of \mathfrak{cls} is inner, i.e., $\text{CDer}(\mathfrak{cls}) = \text{CInn}(\mathfrak{cls})$.*

Proof. Take any $D \in \text{CDer}(\mathfrak{cls})$. For this fixed superderivation D and any $c \in \mathbb{Z}$, define $D^c(x_j) = \pi_{c+j}D(x_j)$ for any $j \in \mathbb{Z}$ and $x_j \in (\mathcal{CL})_j$, where π_c is the natural projection from $\mathbb{C}[\lambda] \otimes \mathfrak{cls}$ to $\mathbb{C}[\lambda] \otimes (\mathfrak{cls})_c$. Then it is easy to check that D^c is a conformal superderivation of \mathfrak{cls} .

We assert that each D^c is inner. For this, we only need to consider the case that D^c is \mathbb{Z}_2 -homogenous.

Case 1. $D^c \in \text{CDer}^{\bar{0}}(\mathfrak{cls})$.

In this case, assume that

$$D_\lambda^c(L_i) = f_i(\partial, \lambda)L_{i+c} \quad \text{and} \quad D_\lambda^c(G_i) = g_i(\partial, \lambda)G_{i+c}$$

for some $f(\partial, \lambda), g_i(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$.

Applying D_λ^c to $[L_0 \mu L_i] = (\partial + 2\mu)L_i$ and comparing the coefficients of L_{i+c} give

$$(\partial + \lambda + 2\mu)f_i(\partial, \lambda) = (\partial + 2\lambda + 2\mu)f_0(-\lambda - \mu, \lambda) + (\partial + 2\mu)f_i(\partial + \mu, \lambda)$$

Setting $\mu = 0$ in the formula above, we get

$$f_i(\partial, i) = (\partial + 2\lambda)\frac{f_0(-\lambda, \lambda)}{\lambda}.$$

Similarly, it follows from $[L_0 \mu G_j] = (\partial + \frac{3}{2}\mu)G_j$ that

$$(\partial + \lambda + \frac{3}{2}\mu)g_j(\partial, \lambda) = (\partial + \frac{3(\lambda + \mu)}{2})f_0(-\lambda - \mu, \lambda) + (\partial + \frac{3\mu}{2})g_j(\partial + \mu, \lambda). \quad (3.1)$$

Setting $\mu = 0$ in (3.1), one has

$$g_j(\partial, \lambda) = (\partial + \frac{3\lambda}{2})\frac{f_0(-\lambda, \lambda)}{\lambda}.$$

Thus, $D_\lambda^c = \text{ad}_{\frac{f_0(\partial, -\partial)}{-\partial} L_c}$.

Case 2. $D^c \in \text{CDer}^1(\mathfrak{cls})$.

Assume that $D_\lambda^c(L_i) = g_i(\partial, \lambda)G_{i+c}$, $D_\lambda^c(G_i) = f_i(\partial, \lambda)L_{i+c}$. It follows from applying D_λ^c to $[L_0 \mu L_i] = (\partial + 2\mu)L_i$ that

$$(\partial + \lambda + 2\mu)g_i(\partial, \lambda) = g_0(-\lambda - \mu, \lambda)\left(\frac{1}{2}\partial + \frac{3}{2}(\lambda + \mu)\right)G_{i+c} + g_i(\partial + \mu, \lambda)\left(\partial + \frac{3\mu}{2}\right),$$

from which by setting $\mu = 0$ one has

$$g_i(\partial, \lambda) = \left(\frac{1}{2}\partial + \frac{3}{2}\lambda\right)\frac{g_0(-\lambda, \lambda)}{\lambda}.$$

Using $[L_0 \mu G_i] = (\partial + \frac{3}{2}\mu)G_i$, one has

$$\left(\partial + \lambda + \frac{3\mu}{2}\right)f_i(\partial, \lambda) = \left(2g_0(-\lambda - \mu, \lambda) - f_i(\partial + \mu, \lambda)(\partial + 2\mu)\right).$$

from which by choosing $\mu = 0$ gives

$$f_i(\partial, \lambda) = \frac{2g_0(-\lambda, \lambda)}{\lambda}.$$

Whence one can see that

$$D_\lambda^c = \text{ad}_{\frac{g_0(\partial, -\partial)}{-\partial}G_c}.$$

So in either case, we see that $D^c = \text{ad}_{x_c}$ for some $x_c \in (\mathfrak{cls})_c$, is inner, completing the assertion. Note from the definition of D^c we see that $D = \sum_{c \in \mathbb{Z}} D^c$. In particular,

$$D(L_0) = \sum_{c \in \mathbb{Z}} \text{ad}_{x_c}(L_0) = \sum_{c \in \mathbb{Z}, x_c \neq 0} \text{ad}_{x_c}(L_0) = \sum_{c \in \mathbb{Z}, x_c \neq 0} \text{ad}_{x_c}(L_0),$$

which must be a finite sum by the fact that D is a linear map from $\oplus_{i \in \mathbb{Z}} (\mathfrak{cls})_i$ to $\oplus_{i \in \mathbb{Z}} (\mathfrak{cls})_i[\lambda]$. Now it follows from the fact $0 \neq \text{ad}_{y_c}(L_0) \in (\mathfrak{cls})_c$ for any $0 \neq y_c \in (\mathfrak{cls})_c$ that all but finitely many x_c are zero, and therefore $\sum_{c \in \mathbb{Z}} x_c \in \mathfrak{cls}$. This implies $D = \sum_{c \in \mathbb{Z}} \text{ad}_{x_c} = \text{ad}_{\sum_{c \in \mathbb{Z}} x_c}$ is an inner conformal superderivation. \square

4 Free modules of rank ≤ 2

Let $V = \mathbb{C}[\partial]x \oplus \mathbb{C}[\partial]y$ be a free $\mathbb{C}[\partial]$ -module of rank two with $V^{\bar{0}} = \mathbb{C}[\partial]x$ and $V^{\bar{1}} = \mathbb{C}[\partial]y$. For any $a, b \in \mathbb{C}$ and $c \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, on the one hand, define the actions of L_i and G_i on V as follows:

$$\begin{aligned} L_i \lambda x &= c^i(\partial + a\lambda + b)x, \quad L_i \lambda y = c^i\left(\partial + \left(a + \frac{1}{2}\right)\lambda + b\right)y, \\ G_i \lambda x &= c^i y, \quad G_i \lambda y = c^i(\partial + 2a\lambda + b)x; \end{aligned} \tag{4.1}$$

on the other hand, actions are given by in another way:

$$\begin{aligned} L_{i\ \lambda} x &= c^i(\partial + a\lambda + b)x, \quad L_{i\ \lambda} y = c^i(\partial + (a - \frac{1}{2})\lambda + b)y, \\ G_{i\ \lambda} x &= c^i(\partial + (2a - 1)\lambda + b)y, \quad G_{i\ \lambda} y = c^i x. \end{aligned} \quad (4.2)$$

It is not hard to see that these two different actions can be extended to the whole \mathfrak{cls} such that in both cases V is a \mathfrak{cls} -module. Let us denote the former \mathfrak{cls} -module by $M_{a,b,c}$ and the latter by $M'_{a,b,c}$.

Proposition 4.1. (i) *The \mathfrak{cls} -module $M_{a,b,c}$ is irreducible if and only if $a \neq 0$, and $M'_{a,b,c}$ is irreducible if and only if $a \neq \frac{1}{2}$. Moreover, $\mathbb{C}[\partial](\partial + b)x \oplus \mathbb{C}[\partial]y$ and $\mathbb{C}[\partial]x \oplus \mathbb{C}[\partial](\partial + b)y$ are the unique nontrivial \mathfrak{cls} -submodules of $M_{0,b,c}$ and $M'_{\frac{1}{2},b,c}$, respectively.*

(ii) *For any $R, T \in \{M, M'\}$, then $R_{a,b,c} \cong T_{\alpha,\beta,\gamma}$ if and only if $(a, b, c) = (\alpha, \beta, \gamma)$ and $R = T$.*

Proof. (i) We only restrict ourself to the irreducibility of $M_{a,b,c}$, the other one can be treated similarly. It is clear that $\mathbb{C}[\partial](\partial + b)x \oplus \mathbb{C}[\partial]y$ is the unique maximal submodule of $M_{0,b,c}$. So the irreducibility of $M_{a,b,c}$ implies that $a \neq 0$.

Conversely, we show that $M_{a,b,c}$ is irreducible if $a \neq 0$. This is equivalent to showing that any submodule $I_{f(\partial),g(\partial)}$ generated by the single nonzero element $f(\partial)x + g(\partial)y$ is the entire $M_{a,b,c}$. Note by applying the action of G_i on this element if necessary we may assume that $f(\partial) \neq 0$. Due to the irreducibility of $\mathbb{C}[\partial]x$ as an \mathfrak{clv} -module (see [8, 16]), we can obtain that

$$x + h(\partial)y \in I_{f(\partial),g(\partial)}$$

for some $h(\partial) \in \mathbb{C}[\partial]$ by applying the actions of \mathfrak{clv} . So $c_i^{-1}G_{i\ \lambda}(x + h(\partial)y) = y + h(\partial + \lambda)(\partial + 2a\lambda + b)x \in I_{f(\partial),g(\partial)}[\lambda]$, and then

$$y + h(\partial)(\partial + b)x \in I_{f(\partial),g(\partial)}.$$

Now these two elements give $0 \neq x - (h(\partial))^2(\partial + b)x \in I_{f(\partial),g(\partial)}$. It follows from the irreducibility of $\mathbb{C}[\partial]x$ as an \mathfrak{clv} -module again that $x \in I_{f(\partial),g(\partial)}$ and therefore $y \in I_{f(\partial),g(\partial)}$ by (4.1). Hence, $I_{f(\partial),g(\partial)} = M_{a,b,c}$, completing the proof.

(ii) Assume that $R_{a,b,c} = \mathbb{C}[\partial]x \oplus \mathbb{C}[\partial]y$ and $T_{\alpha,\beta,\gamma} = \mathbb{C}[\partial]x' \oplus \mathbb{C}[\partial]y'$, and let

$$\phi : R_{a,b,c} \rightarrow T_{\alpha,\beta,\gamma}$$

be an isomorphism. Then there exist some $f(\partial), g(\partial) \in \mathbb{C}[\partial]$ such that

$$\phi(x) = f(\partial)x' \quad \text{and} \quad \phi(y) = g(\partial)y'.$$

It follows immediately from $\phi(L_{i\lambda}x) = L_{i\lambda}\phi(x)$ that

$$c^i(\partial + a\lambda + b)f(\partial)x' = \gamma^i f(\partial + \lambda)(\partial + \alpha\lambda + \beta)x',$$

from which we can see that $(a, b, c) = (\alpha, \beta, \gamma)$ and that $f(\partial)$ is a constant term, which may be assumed to be 1 by means of replacing x' by $f(\partial)x'$. It remains to show that $R = T$. Suppose on the contrary that $R \neq T$. Without loss of generality, we assume that $R = M$ and $T = M'$. Then from $\phi(G_{i\lambda}x) = G_{i\lambda}\phi(x)$ we obtain that $a = \frac{1}{2}$. By (i), $M_{\frac{1}{2},b,c}$ is irreducible but $M'_{\frac{1}{2},b,c}$ is reducible, so ϕ can not be isomorphic, a contradiction. \square

The aim of this section is to classify all free \mathbf{cls} -modules of rank two. In fact, the two classes of \mathbf{cls} -modules constructed as above exhaust all free \mathbf{cls} -modules of rank two.

Theorem 4.2. *Suppose that $V = V^{\bar{0}} \oplus V^{\bar{1}}$ is a nontrivial free \mathbf{cls} -module of rank two. Then V is either isomorphism to $M_{a,b,c}$ defined by (4.1) or $M'_{a,b,c}$ defined by (4.2).*

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a nontrivial free \mathbf{cls} -module of rank two. Then both $V^{\bar{0}}$ and $V^{\bar{1}}$ are nontrivial. To see this, suppose on the contrary that $V^{\bar{i}} = 0$, then $G_{j\lambda}V^{\bar{i}+1} = 0$ and hence $L_{j\lambda}V^{\bar{i}+1} = 0$ for any $j \in \mathbb{Z}$. So in this case V is a trivial \mathbf{cls} -module, a contradiction. Hence both $V^{\bar{0}} = \mathbb{C}[\partial]x$ and $V^{\bar{1}} = \mathbb{C}[\partial]y$ are $\mathbb{C}[\partial]$ and hence \mathbf{cls} -modules of rank one.

Observe that V cannot be a trivial \mathbf{cls} -module, since the relations $[L_{i\lambda}G_j] = (\partial + \frac{3}{2}\lambda)G_{i+j}$ for any $i, j \in \mathbb{Z}$ would imply the actions of G_i on V are all trivial. Now it follows from [16, Proposition 4.3] that there exist $a, a', b, b', c, d \in \mathbb{C}$ such that

$$L_{i\lambda}x = c^i(\partial + a\lambda + b)x, \quad L_{i\lambda}y = d^i(\partial + a'\lambda + b')y, \quad \forall i \in \mathbb{Z}.$$

Note that at least one of c and d is nonzero, without loss of generality, we assume that $c \neq 0$.

In order to determine the module structure of \mathbf{cls} on V we only need to give the explicit actions of G_i on x and y . Assume

$$G_{i\lambda}x = c^i g_i(\partial, \lambda)y \quad \text{and} \quad G_{i\lambda}y = c^i g'_i(\partial, \lambda)x \quad \text{for some } g_i(\partial, \lambda), g'_i(\partial, \lambda) \in \mathbb{C}[\partial, \lambda].$$

It follows from $[G_{i\lambda}G_j]_{\lambda+\mu}x = 2L_{i+j\lambda+\mu}x$ that

$$g_i(\partial + \lambda, \mu)g'_j(\partial, \lambda) + g_i(\partial + \mu, \lambda)g'_j(\partial, \mu) = \partial + a(\lambda + \mu) + b, \quad \forall i, j \in \mathbb{Z}. \quad (4.3)$$

In particular, setting $\lambda = \mu$ and $i = j$ in above formula one has

$$g_i(\partial + \lambda, \lambda)g'_i(\partial, \lambda) = \partial + 2a\lambda + b, \quad (4.4)$$

which immediately implies

$$g_i(\partial, \lambda) \neq 0, \quad g'_i(\partial, \lambda) \neq 0, \quad \forall i \in \mathbb{Z}. \quad (4.5)$$

This together with $[G_{i\lambda}G_j]_{\lambda+\mu}y = 2L_{i+j\lambda+\mu}y$ forces $d \neq 0$. So we have arrived at the following lemma.

Lemma 4.3. *The polynomials $g_i(\partial, \lambda), g'_i(\partial, \lambda)$ for all $i \in \mathbb{Z}$ and complex numbers c, d are all nonzero.*

Using this lemma, we now can give the proof of the theorem above.

Proof of Theorem 4.2 Note that for any $i \in \mathbb{Z}$ by (4.4) either $\deg_{\partial} g_i(\partial, \lambda) = 1$ and $\deg_{\partial} g'_i(\partial, \lambda) = 0$ or $\deg_{\partial} g_i(\partial, \lambda) = 0$ and $\deg_{\partial} g'_i(\partial, \lambda) = 1$.

Case 1. *There exists some $i \in \mathbb{Z}$ such that $\deg_{\partial} g_i(\partial, \lambda) = 1$ and $\deg_{\partial} g'_i(\partial, \lambda) = 0$.*

It follows from $[L_j \lambda G_i]_{\lambda+\mu} x = (\partial + \frac{3}{2}\lambda) G_{j+i} \lambda + \mu x$ for any $i, j \in \mathbb{Z}$ that

$$g_i(\partial + \lambda, \mu)(d/c)^j(\partial + a'\lambda + b') - g_i(\partial, \mu)(\partial + \mu + a\lambda + b) = (\frac{1}{2}\lambda - \mu)g_{j+i}(\partial, \lambda + \mu). \quad (4.6)$$

Now view terms on both sides of (4.6) as polynomials in the variable ∂ with coefficients in $\mathbb{C}[\lambda, \mu]$, then the coefficient of ∂^2 in the left hand side must be zero, since $\deg_{\partial} g_{i+j}(\partial, \lambda + \mu) \leq 1$. This forces $d = c$ and hence (4.6) turns out to be

$$g_i(\partial + \lambda, \mu)(\partial + a'\lambda + b') - g_i(\partial, \mu)(\partial + \mu + a\lambda + b) = (\frac{1}{2}\lambda - \mu)g_{j+i}(\partial, \lambda + \mu). \quad (4.7)$$

Furthermore, by choosing $\lambda = 0$ in (4.7) and comparing the coefficients of $\partial\mu$ one can see that $\deg_{\partial} g_j(\partial, \lambda) = 1$ for all $j \in \mathbb{Z}$. This allows us for each $i \in \mathbb{Z}$ to write

$$g_i(\partial, \lambda) = \partial s_i(\lambda) + t_i(\lambda) \quad \text{for some } s_i(\lambda), t_i(\lambda) \in \mathbb{C}[\lambda].$$

Using these expressions, (4.7) is equivalent to

$$s_i(\mu)((a' - a)\lambda + b' - b - \mu) + \lambda s_i(\mu) = (\frac{1}{2}\lambda - \mu)s_{i+j}(\lambda + \mu) \quad (4.8)$$

$$\text{and } t_i(\mu)((a' - a)\lambda + b' - b - \mu) + \lambda(a'\lambda + b')s_i(\mu) = (\frac{1}{2}\lambda - \mu)t_{i+j}(\lambda + \mu). \quad (4.9)$$

It is not hard to observe from (4.8) that all $s_i(\lambda)$ are equal to a nonzero constant γ , and also that $a' = a - \frac{1}{2}$, $b' = b$. Moreover, replacing y by γy we may assume that $\gamma = 1$. At this very moment, (4.9) can be rewritten as

$$-(\frac{1}{2}\lambda + \mu)t_i(\mu) + (a - \frac{1}{2})\lambda^2 + b\lambda = (\frac{1}{2}\lambda - \mu)t_{i+j}(\lambda + \mu), \quad (4.10)$$

from which one can see that $\deg t_i(\lambda) \leq 1$ and thus has the form $t_i(\lambda) = \alpha_i\lambda + \beta_i$ for any $i \in \mathbb{Z}$ and some $\alpha_i, \beta_i \in \mathbb{C}$. Substituting these explicit expressions into (4.10) and carrying a direct computation show that $t_i(\lambda) = (2a - 1)\lambda + b$ for any $i \in \mathbb{Z}$.

To sum up, so far under the assumption

$$L_i \lambda x = c^i(\partial + a\lambda + b)x$$

we have obtained the following actions

$$L_{i\lambda}y = c^i(\partial + (a - \frac{1}{2})\lambda + b)y \quad \text{and} \quad G_{i\lambda}x = c^i(\partial + (2a - 1)\lambda + b)y.$$

Then by the remark at the very beginning of this proof, $g'_i(\lambda) := g'_i(\partial, \lambda) \in \mathbb{C}[\lambda]$. It follows from $[G_{i\lambda}G_j]_{\lambda+\mu}y = 2L_{i+j\lambda+\mu}y$ that

$$(\partial + (2a - 1)\lambda + b)g'_j(\mu) + (\partial + (2a - 1)\mu + b)g'_i(\lambda) = 2(\partial + (a - \frac{1}{2})(\lambda + \mu) + b),$$

which gives immediately rise to $g'_i(\lambda) = 1$ for all $i \in \mathbb{Z}$. Whence $V \cong M'_{a,b,c}$.

Case 2. *There exists some $i \in \mathbb{Z}$ such that $\deg_{\partial} g_i(\partial, \lambda) = 0$ and $\deg_{\partial} g'_i(\partial, \lambda) = 1$.*

For this case, similar arguments as for the previous one can be adopted to show that $V \cong M_{a,b,c}$. \square

We conclude this section with the following classification result, which is a consequence of Proposition 4.1 and Theorem 4.2.

Theorem 4.4. (i) *The set $\{M_{a,b,c}, M'_{\alpha,\beta,\gamma} \mid a, b, \alpha, \beta, \gamma \in \mathbb{C} \text{ and } c, \gamma \in \mathbb{C}^* \text{ with } a \neq 0, \alpha \neq \frac{1}{2}\}$ is a complete list of inequivalent irreducible \mathfrak{cls} -modules of rank two.*

(ii) *Any \mathfrak{cls} -module V generated by two elements is either trivial or one of the two \mathfrak{cls} -modules $M_{a,b,c}$ and $M'_{a,b,c}$ for some $a, b, c \in \mathbb{C}$.*

Proof. (i) is obvious. (ii) Since any $\mathbb{C}[\partial]$ -module generated by two elements is a quotient of some free $\mathbb{C}[\partial]$ -module of rank two, V is a quotient of $M_{a,b,c}$ or $M'_{a,b,c}$ for some $a, b, c \in \mathbb{C}$ by Theorem 4.2. Then (ii) follows from the fact that the nontrivial quotient of $M_{0,b,c}$ or $M'_{\frac{1}{2},b,c}$ is a trivial \mathfrak{cls} -module by Proposition 4.1 (i). \square

5 Free \mathbb{Z} -graded modules of rank ≤ 2

Let us first collect some results concerning nontrivial free \mathbb{Z} -graded \mathfrak{clv} -modules of rank one. Two classes of such modules were introduced on the free $\mathbb{C}[\partial]$ -module $\oplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]v_i$ in [16]:

(i) Denote by $V_{a,b}$ when $\oplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]v_i$ is equipped with the \mathfrak{clv} -module structure given by

$$L_{i\lambda}v_j = (\partial + a\lambda + b)v_{i+j}, \quad \forall i \in \mathbb{Z},$$

where $a, b \in \mathbb{C}$;

(ii) Denote by $V_{A,b}$ when $\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]v_i$ is equipped with the \mathfrak{clb} -module structure given by

$$L_i \lambda v_j = \begin{cases} (\partial + b)v_{i+j}, & \text{if } (a_j, a_{i+j}) = (0, 0), \\ (\partial + \lambda + b)v_{i+j}, & \text{if } (a_j, a_{i+j}) = (1, 1), \\ v_{i+j}, & \text{if } (a_j, a_{i+j}) = (0, 1), \\ (\partial + b)(\partial + \lambda + b)v_{i+j}, & \text{if } (a_j, a_{i+j}) = (1, 0), \end{cases}$$

where $A = (a_i)_{i \in \mathbb{Z}}$ is an element of the product $\prod_{i \in \mathbb{Z}} \mathbb{Z}_2$ of $\{\mathbb{Z}_2\}_{i \in \mathbb{Z}}$. Note that $V_{\{0\}_{i \in \mathbb{Z}}, b}$ coincides with $V_{0,b}$ and $V_{\{1\}_{i \in \mathbb{Z}}, b}$ coincides with $V_{1,b}$.

We cite the classification result of nontrivial free \mathbb{Z} -graded \mathfrak{clb} -modules of rank one from [16] as a lemma here.

Lemma 5.1. *Suppose that V is a free \mathbb{Z} -graded \mathfrak{clb} -module of rank one. Then $V = V_{a,b}$ or $V = V_{A,b}$ for some $a, b \in \mathbb{C}$ and $A \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2$.*

Let $\{x_i \mid i \in \mathbb{Z}\}$ and $\{y_i \mid i \in \mathbb{Z}\}$ be two $\mathbb{C}[\partial]$ -linearly independent sets, and form the $\mathbb{C}[\partial]$ -module $V = \bigoplus_{i \in \mathbb{Z}} (\mathbb{C}[\partial]x_i \oplus \mathbb{C}[\partial]y_i)$. For any $a, b \in \mathbb{C}$ and $A \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2$, we next give four different actions of \mathfrak{cls} on V such that $\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]x_i$ and $\bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]y_i$ have the form $V_{a,b}$ or $V_{A,b}$ as \mathfrak{clb} -modules:

$$\begin{aligned} \text{(i)} \quad & L_i \lambda x_j = (\partial + a\lambda + b)x_{i+j}, & L_i \lambda y_j &= (\partial + (a - \frac{1}{2})\lambda + b)y_{i+j}, \\ & G_i \lambda x_j = (\partial + (2a - 1)\lambda + b)y_{i+j}, & G_i \lambda y_j &= x_{i+j}; \\ \text{(ii)} \quad & L_i \lambda x_j = (\partial + a\lambda + b)x_{i+j}, & L_i \lambda y_j &= (\partial + (a + \frac{1}{2})\lambda + b)y_{i+j}, \\ & G_i \lambda x_j = y_{i+j}, & G_i \lambda y_j &= (\partial + 2a\lambda + b)x_{i+j}; \\ \text{(iii)} \quad & L_i \lambda x_j = (\partial + \frac{1}{2}\lambda + b)x_{i+j} \quad (\text{the case } a = \frac{1}{2}); \end{aligned}$$

$$\begin{aligned} L_i \lambda y_j &= \begin{cases} (\partial + b)y_{i+j}, & \text{if } (a_j, a_{i+j}) = (0, 0), \\ (\partial + \lambda + b)y_{i+j}, & \text{if } (a_j, a_{i+j}) = (1, 1), \\ y_{i+j}, & \text{if } (a_j, a_{i+j}) = (0, 1), \\ (\partial + b)(\partial + \lambda + b)y_{i+j}, & \text{if } (a_j, a_{i+j}) = (1, 0); \end{cases} \\ G_i \lambda x_j &= \begin{cases} (\partial + b)y_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(0, 0), (1, 0)\}, \\ y_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(1, 1), (0, 1)\}; \end{cases} \\ G_i \lambda y_j &= \begin{cases} (\partial + \lambda + b)x_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(1, 1), (1, 0)\}, \\ x_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(0, 0), (0, 1)\}. \end{cases} \end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad L_{i \lambda} x_j &= \begin{cases} (\partial + b)x_{i+j}, & \text{if } (a_j, a_{i+j}) = (0, 0), \\ (\partial + \lambda + b)x_{i+j}, & \text{if } (a_j, a_{i+j}) = (1, 1), \\ x_{i+j}, & \text{if } (a_j, a_{i+j}) = (0, 1), \\ (\partial + b)(\partial + \lambda + b)x_{i+j}, & \text{if } (a_j, a_{i+j}) = (1, 0); \end{cases} \\
L_{i \lambda} y_j &= (\partial + \frac{1}{2}\lambda + b)y_{i+j} \quad (\text{the case } a = \frac{1}{2}); \\
G_{i \lambda} x_j &= \begin{cases} (\partial + \lambda + b)y_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(1, 1), (1, 0)\}, \\ y_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(0, 0), (0, 1)\}; \end{cases} \\
G_{i \lambda} y_j &= \begin{cases} (\partial + b)x_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(0, 0), (1, 0)\}, \\ x_{i+j}, & \text{if } (a_j, a_{i+j}) \in \{(1, 1), (0, 1)\}; \end{cases}
\end{aligned}$$

Denote V by $M_{a,b}$, $M'_{a,b}$, $M_{A,b}$, $M'_{A,b}$ in the cases (i)-(iv), respectively. One can verify under the given actions, $M_{a,b}$, $M'_{a,b}$, $M_{A,b}$ and $M'_{A,b}$ becomes \mathbf{cls} -modules. Note that they are all \mathbb{Z}_2 -graded with the even part $\oplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]x_i$ and the odd part $\oplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]y_i$, and also \mathbb{Z} -graded with $\mathbb{C}[\partial]x_i \oplus \mathbb{C}[\partial]y_i$ as their i -th gradation. So the four classes of modules above are all free \mathbb{Z} -graded \mathbf{cls} -modules of rank two.

Theorem 5.2. *Suppose that $V = V_0 \oplus V_1$ is a nontrivial free \mathbb{Z} -graded \mathbf{cls} -module of rank two. Then V is isomorphic to one of the following modules:*

$$M_{a,b}, M'_{a,b}, M_{A,b}, M'_{A,b}, \quad \text{where } a, b \in \mathbb{C} \text{ and } A \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2.$$

Proof. Note that both $V^{\bar{0}}$ and $V^{\bar{1}}$ are nontrivial by the remark after Theorem 4.2, hence are nontrivial free \mathbb{Z} -graded \mathbf{clb} -modules. Let $\{x_i \mid i \in \mathbb{Z}\}$ and $\{y_i \mid i \in \mathbb{Z}\}$ be $\mathbb{C}[\partial]$ -bases of V_0 and V_1 . Then by Lemma 5.1, $V^{\bar{0}} = V_{a,b}$ or $V^{\bar{0}} = V_{A,b}$ for some $a, b \in \mathbb{C}$ and $A \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2$. Assume that

$$\begin{aligned}
L_{i \lambda} x_j &= f_{i,j}(\partial, \lambda)x_{i+j}, & G_{i \lambda} x_j &= g_{i,j}(\partial, \lambda)y_{i+j}, \\
L_{i \lambda} y_j &= f'_{i,j}(\partial, \lambda)y_{i+j}, & G_{i \lambda} y_j &= g'_{i,j}(\partial, \lambda)x_{i+j}
\end{aligned}$$

for any $i, j, k \in \mathbb{Z}$ and some $f(\partial, \lambda), f'(\partial, \lambda), g(\partial, \lambda), g'(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$. It follows from $[L_{i \lambda} G_j]_{\lambda+\mu} x_k = (\partial + \frac{3}{2}\lambda)G_{i+j \lambda+\mu} x_k$ that

$$\begin{aligned}
&g_{j,k}(\partial + \lambda, \mu)f'_{i,j+k}(\partial, \lambda) - g_{j,i+k}(\partial, \mu)f_{i,k}(\partial + \mu, \lambda) \\
&= \left(\frac{1}{2}\lambda - \mu\right)g_{i+j,k}(\partial, \lambda + \mu), \quad \forall i, j, k \in \mathbb{Z}.
\end{aligned} \tag{5.1}$$

Claim 1. *If $V^{\bar{0}} = V_{a,b}$ as \mathbf{clb} -modules, then $V^{\bar{1}} = V_{a-\frac{1}{2},b}$, $V_{a+\frac{1}{2},b}$ or $V_{A,b}$ ($a = \frac{1}{2}$). In this case $V = M_{a,b}$, $M'_{a,b}$ or $M_{A,b}$.*

Assume that $V^{\bar{0}} = V_{a,b}$. This entails us to write explicitly the expressions of $f_{i,j}(\partial, \lambda)$ as:

$$f_{i,j}(\partial, \lambda) = \partial + a\lambda + b.$$

It follows from $[G_i \lambda G_j]_{\lambda+\mu} x_k = 2L_{i+j} \lambda_{\lambda+\mu} x_k$ that

$$g_{j,k}(\partial + \lambda, \mu) g'_{i,j+k}(\partial, \lambda) + g_{i,k}(\partial + \mu, \lambda) g'_{j,i+k}(\partial, \mu) = 2(\partial + a(\lambda + \mu) + b), \quad \forall i, j \in \mathbb{Z}. \quad (5.2)$$

Setting $j = i$ and $\mu = \lambda$ in (5.2) gives

$$g_{i,k}(\partial + \lambda, \lambda) g'_{i,i+k}(\partial, \lambda) = \partial + 2a\lambda + b, \quad \forall i, k \in \mathbb{Z}, \quad (5.3)$$

from which it is not hard to see that for any $i, k \in \mathbb{Z}$,

$$\begin{aligned} & \text{either } g_{i,k}(\partial, \lambda) := \mathfrak{g}_{i,k} \text{ is a constant and } g'_{i,i+k}(\partial, \lambda) = \mathfrak{g}_{i,k}^{-1}(\partial + 2a\lambda + b) \\ & \text{or } g'_{i,i+k}(\partial, \lambda) := \mathfrak{g}_{i,i+k} \text{ is a constant and } g_{i,k}(\partial, \lambda) = \mathfrak{g}_{i,i+k}^{-1}(\partial + (2a - 1)\lambda + b). \end{aligned} \quad (5.4)$$

Case 1. As \mathfrak{clv} -modules, $V^{\bar{1}} = V_{A,\beta}$ for some $A \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2 \setminus \{\{0\}_{i \in \mathbb{Z}}, \{1\}_{i \in \mathbb{Z}}\}$ and $\beta \in \mathbb{C}$. Then $a = \frac{1}{2}, \beta = b$ and $V = M_{A,b}$.

Note that there are four different expressions for $f'_{i,j}(\partial, \lambda)$. We will check it case by case.

(a) There exist some $p, q \in \mathbb{Z}$ such that $f'_{p,q}(\partial, \lambda) = (\partial + \beta)(\partial + \lambda + \beta)$.

Choosing $i = p, k = q - j$ and using $f_{i,k}(\partial, \lambda) = \partial + a\lambda + b$ in (5.1) we have

$$\begin{aligned} & g_{j,q-j}(\partial + \lambda, \mu)(\partial + \beta)(\partial + \lambda + \beta) - g_{j,p+q-j}(\partial, \mu)(\partial + \mu + a\lambda + b) \\ &= \left(\frac{1}{2}\lambda - \mu\right) g_{p+j,q-j}(\partial, \lambda + \mu), \quad \forall j \in \mathbb{Z}. \end{aligned} \quad (5.5)$$

Combining this with (5.4), then we must have

$$\begin{aligned} & \beta = b, a = \frac{1}{2}, \mathfrak{g}_{j,q-j} = g_{j,q-j}(\partial + \lambda, \mu) \in \mathbb{C}^*, \\ & g_{j,p+q-j}(\partial, \mu) = \mathfrak{g}_{j,q-j}(\partial + b) \text{ and } g_{p+j,q-j}(\partial, \lambda + \mu) = \mathfrak{g}_{j,q-j}(\partial + b). \end{aligned}$$

It can be also observed from (5.5) that all the leading coefficients of $g_{k,l}(\partial, \lambda)$ are equal (here we regard $g_{k,l}(\partial, \lambda)$ as polynomials in ∂ with coefficients in $\mathbb{C}[\lambda]$). By rescaling the basis $\{y_i \mid i \in \mathbb{Z}\}$ we may assume that all the leading coefficients of $g_{k,l}(\partial, \lambda)$ are equal 1. Hence,

$$g_{j,q-j}(\partial, \lambda) = 1 \quad \text{and} \quad g_{j,p+q-j}(\partial, \lambda) = \partial + b, \quad \forall j \in \mathbb{Z},$$

which in turn by using (5.4) forces

$$g'_{j,q}(\partial, \lambda) = \partial + \lambda + b \quad \text{and} \quad g'_{j,p+q}(\partial, \lambda) = 1, \quad \forall j \in \mathbb{Z}.$$

In particular,

$$g_{p,q}(\partial, \lambda) = \partial + b \quad \text{and} \quad g'_{p,q}(\partial, \lambda) = \partial + \lambda + b.$$

Similarly, for any $j \in \mathbb{Z}$ we have:

- (b) $g_{p,q}(\partial, \lambda) = \partial + b, g'_{p,q}(\partial, \lambda) = 1$ if $f'_{p,q}(\partial, \lambda) = \partial + b$;
- (c) $g_{p,q}(\partial, \lambda) = 1, g'_{p,q}(\partial, \lambda) = \partial + \lambda + b$ if $f'_{p,q}(\partial, \lambda) = \partial + \lambda + b$;
- (d) $g_{p,q}(\partial, \lambda) = g'_{p,q}(\partial, \lambda) = 1$ if $f'_{p,q}(\partial, \lambda) = 1$.

This is exactly the module $M_{A,b}$, completing Case 1.

Case 2. As \mathfrak{clv} -modules $V^{\bar{1}} = V_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{C}$. Then $\alpha = a \pm \frac{1}{2}$, $\beta = b$ and $V = M_{a,b}$ or $M'_{a,b}$.

In this case $f'_{i,j}(\partial, \lambda) = \partial + \alpha\lambda + \beta$. So (5.1) now becomes

$$\begin{aligned} & g_{j,k}(\partial + \lambda, \mu)(\partial + \alpha\lambda + \beta) - g_{j,i+k}(\partial, \mu)(\partial + \mu + a\lambda + b) \\ &= \left(\frac{1}{2}\lambda - \mu\right)g_{i+j,k}(\partial, \lambda + \mu), \forall i, j, k \in \mathbb{Z}. \end{aligned}$$

Observe from this formula that the coefficients of ∂ in $g_{j,i+k}(\partial + \lambda, \lambda)$, $g_{j,k}(\partial, \mu)$ and $g_{i+j,k}(\partial, \lambda + \mu)$ are equal, which may be assumed to be 1 for convenience. Moreover, the degree of $g_{j,k}(\partial, \lambda)$ for any $j, k \in \mathbb{Z}$ is a constant, i.e, $\deg_{\partial} g_{j,k}(\partial, \lambda) = 1$ for all $j, k \in \mathbb{Z}$ or $\deg_{\partial} g_{j,k}(\partial, \lambda) = 0$ for all $j, k \in \mathbb{Z}$. In the former case,

$$g_{j,k}(\partial, \lambda) = \partial + (2a - 1)\lambda + b, g'_{j,k}(\partial, \lambda) = 1, \beta = b \quad \text{and} \quad \alpha = a - \frac{1}{2},$$

that is, $V = M_{a,b}$; while in the later case,

$$g_{j,k}(\partial, \lambda) = 1, g'_{j,k}(\partial, \lambda) = \partial + 2a\lambda + b, \beta = b \quad \text{and} \quad \alpha = a + \frac{1}{2},$$

that is, $V = M'_{a,b}$.

Claim 2. The case $V^{\bar{0}} = V_{A,b}$ and $V^{\bar{1}} = V_{B,\beta}$ for some $A, B \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2 \setminus \{\{0\}_{i \in \mathbb{Z}}, \{1\}_{i \in \mathbb{Z}}\}$ and $b, \beta \in \mathbb{C}$ can not occur.

Suppose on the contrary that $V^{\bar{0}} = V_{A,b}$ and $V^{\bar{1}} = V_{B,\beta}$. Choosing $i = \mu = 0$ in (5.1), one has

$$g_{j,k}(\partial + \lambda, 0)f'_{0,k}(\partial, \lambda) - g_{j,k}(\partial, 0)f_{0,k}(\partial, \lambda) = \frac{1}{2}\lambda g_{j,k}(\partial, \lambda).$$

While by definition of the actions of \mathfrak{clv} ,

$$f_{0,k}(\partial, \lambda) = \partial + r_k\lambda + b, \quad f'_{0,k}(\partial, \lambda) = \partial + t_k\lambda + \beta, \quad \text{where } r_k, t_k \in \mathbb{Z}_2.$$

Combining these two formulae gives

$$g_{j,k}(\partial + \lambda, 0)(\partial + t_k \lambda + b') - g_{j,k}(\partial, 0)(\partial + r_k \lambda + b) = \frac{1}{2} \lambda g_{j,k}(\partial, \lambda). \quad (5.6)$$

Then $b' = b$, by choosing $\lambda = 0$. Now we put (5.6) in the following form:

$$g_{j,k}(\partial, \lambda) = 2(\partial + b) \frac{g_{j,k}(\partial + \lambda, 0) - g_{j,k}(\partial, 0)}{\lambda} + 2r g_{j,k}(\partial + \lambda, 0) - 2t g_{j,k}(\partial, 0).$$

Taking the operation $\lim_{\lambda \rightarrow 0}$ on both sides of the above formula we see that $g_{j,k}(\partial, 0)$ is solutions of the ordinary differential equation

$$2(\partial + b) \frac{d\phi}{d\partial} = (1 - 2(r_k - t_k))\phi.$$

It is well-known that the general solution of this differential equation is of the form $c(\partial + b)^{\frac{1-2(r_k-t_k)}{2}}$ with $c \in \mathbb{C}$. Thus $g_{j,k}(\partial, 0) \in \mathbb{C}[\partial] \cap \mathbb{C}(\partial + b)^{\frac{1-2(r_k-t_k)}{2}} = 0$, since $\frac{1-2(r_k-t_k)}{2}$ is not an integer. Hence, $g_{j,k}(\partial, 0) = 0$ and thereby $g_{j,k}(\partial, \lambda) = 0$ by (5.6), a contradiction.

Claim 3. *If $V^{\bar{0}} = V_{A,b}$ for some $A \in \prod_{i \in \mathbb{Z}} \mathbb{Z}_2 \setminus \{\{0\}_{i \in \mathbb{Z}}, \{1\}_{i \in \mathbb{Z}}\}$ and $b \in \mathbb{C}$, then $V^{\bar{1}} = V_{\frac{1}{2},b}$ and $V = M'_{A,b}$.*

By Claim 2, $V^{\bar{1}} = V_{\alpha,\beta}$ as \mathbf{cls} -modules for some $\alpha, \beta \in \mathbb{C}$. Now it follows from proof of Case 1 that $V^{\bar{1}} = V_{\frac{1}{2},b}$ and thereby $V = M'_{A,b}$. \square

The irreducible submodules of nontrivial free \mathbb{Z} -graded \mathbf{cls} -modules of rank two can be characterized in the proposition.

Proposition 5.3. *We have the following results:*

(1) *Any nontrivial irreducible \mathbf{cls} -submodule of $M_{a,b}$ has the form*

$$\begin{aligned} & \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial](\partial + b) \sum_{i \in I} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i y_{i+k} \quad \text{if } a = 0, \\ & \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial](\partial + b) \sum_{i \in I} c_i y_{i+k} \quad \text{if } a = \frac{1}{2}, \\ & \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i y_{i+k} \quad \text{if } a \neq 0 \text{ and } a \neq \frac{1}{2}, \end{aligned}$$

where I is a finite subset of \mathbb{Z} and $(c_i)_{i \in I}$ is a sequence of complex numbers;

(2) *Any nontrivial irreducible \mathbf{cls} -submodule of $M'_{a,b}$ has the form*

$$\begin{aligned} & \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial](\partial + b) \sum_{i \in I} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i y_{i+k} \quad \text{if } a = 0, \\ & \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial](\partial + b) \sum_{i \in I} c_i y_{i+k} \quad \text{if } a = -\frac{1}{2}, \\ & \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i y_{i+k} \quad \text{if } a \neq 0 \text{ and } a \neq -\frac{1}{2}, \end{aligned}$$

where I is a finite subset of \mathbb{Z} and $(c_i)_{i \in I}$ is a sequence of complex numbers;

(3) Any nontrivial irreducible \mathbf{cls} -submodule of $M_{A,b}$ has the form

$$\bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i \delta_{i+k}(\partial) y_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i y_{i+k},$$

where I is a finite subset of \mathbb{Z} , $(c_i)_{i \in I}$ is a sequence of complex numbers and

$$\delta_i(\partial) = \begin{cases} (\partial + b), & \text{if } a_i = 0, \\ 1, & \text{if } a_i = 1; \end{cases}$$

(4) Any nontrivial irreducible \mathbf{cls} -submodule of $M'_{A,b}$ has the form

$$\bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i \delta_{i+k}(\partial) x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I} c_i y_{i+k},$$

where I is a finite subset of \mathbb{Z} , $(c_i)_{i \in I}$ is a sequence of complex numbers and $\delta_i(\partial)$ is defined as in (3).

Proof. Here we only give the proofs of (1) and (3). Denote by X and Y the sets $\{x_i \mid i \in \mathbb{Z}\}$ and $\{y_i \mid i \in \mathbb{Z}\}$, respectively. Let $S_{a,b}$ be a nontrivial irreducible \mathbf{cls} -submodule. For any $0 \neq u \in S_{a,b}$, there exists a finite subset $\text{Supp}(u)$ of $X \cup Y$ such that $u \in \bigoplus_{z \in \text{Supp}(u)} \mathbb{C}[\partial] z$. Take $0 \neq u_0 \in S_{a,b}$ with $\#\text{Supp}(u_0)$ minimal. Then we can write $u_0 = \sum_{z \in I_{u_0}} f_z(\partial) z$ for some $f_z(\partial) \in \mathbb{C}[\partial] \setminus \{0\}$.

(1) Consider the case $a \neq 0$. Without loss of generality, we assume that the set $\text{Supp}(u_0) \cap \{x_i \mid i \in \mathbb{Z}\}$ is nonempty. Fix $z' \in X \cap \text{Supp}(u_0)$. By the irreducibility of $\mathbb{C}[\partial] z'$ as $\mathbb{C}[\partial] L_0$ -module [8], we may assume that $f_{z'}(\partial) = 1$. Consider the elements

$$\begin{aligned} u'_0(c_1, c_2) &= L_0 \lambda(u_0)|_{\lambda=c_1} - L_0 \lambda(u_0)|_{\lambda=c_2} - a(c_1 - c_2)u_0 \\ &= \sum_{z' \neq z \in X \cap \text{Supp}(u_0)} \left(f_z(\partial + c_1)(\partial + ac_1 + b) - f_z(\partial + c_2)(\partial + ac_2 + b) - a(c_1 - c_2)f_z(\partial) \right) z \\ &\quad + \sum_{z \in Y \cap \text{Supp}(u_0)} \left(f_z(\partial + c_1)(\partial + (a - \frac{1}{2})c_1 + b) - f_z(\partial + c_2)(\partial + (a - \frac{1}{2})c_2 + b) \right. \\ &\quad \left. - a(c_1 - c_2)f_z(\partial) \right) z \in S_{a,b} \end{aligned}$$

where $c_i \in \mathbb{C}$. Now the minimality of $\text{Supp}(u_0)$ implies that $u'_0(c_1, c_2) = 0$, namely, for any $c_i \in \mathbb{C}$ we have

$$\begin{aligned} 0 &= f_z(\partial + c_1)(\partial + ac_1 + b) - f_z(\partial + c_2)(\partial + ac_2 + b) \\ &\quad - a(c_1 - c_2)f_z(\partial), \quad \forall z \in X \cap \text{Supp}(u_0) \end{aligned}$$

and

$$\begin{aligned} 0 &= f_z(\partial + c_1)(\partial + (a - \frac{1}{2})c_1 + b) - f_z(\partial + c_2)(\partial + (a - \frac{1}{2})c_2 + b) \\ &\quad - a(c_1 - c_2)f_z(\partial), \quad \forall z \in Y \cap \text{Supp}(u_0). \end{aligned}$$

Hence, $0 \neq c_z := f_z(\partial) \in \mathbb{C}$ ($c_{z'} = 1$) for each $z \in X \cap \text{Supp}(u_0)$ and $f_z(\partial) = 0$ for each $z \in Y \cap \text{Supp}(u_0)$. By the minimality of $\text{Supp}(u_0)$ again, $Y \cap \text{Supp}(u_0)$ is empty. That is, $u_0 = \sum_{z \in X \cap \text{Supp}(u_0)} c_z z \in S_{a,b}$. Set $I_{u_0} = \{i \mid x_i \in X \cap \text{Supp}(u_0)\}$. Then u_0 can be written as $u_0 = \sum_{i \in I_{u_0}} c_i x_i$ with $c_i = c_{x_i}$. Now one can see that

$$\begin{aligned} S_{a,b} &= \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I_{u_0}} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial](\partial + b) \sum_{i \in I_{u_0}} c_i y_{i+k} \quad \text{if } a = \frac{1}{2} \\ \text{and } S_{a,b} &= \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I_{u_0}} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I_{u_0}} c_i y_{i+k} \quad \text{if } a \neq \frac{1}{2}. \end{aligned}$$

Let us assume that $a = 0$ and in this case we start with the nonempty set $Y \cap \text{Supp}(u_0)$. Similarly, one can show that

$$S_{a,b} = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial](\partial + b) \sum_{i \in I_{u_0}} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I_{u_0}} c_i y_{i+k}.$$

(3) As for the case $a \neq 0$ in the proof of (1), $u_0 = \sum_{i \in I_{u_0}} c_i x_i \in S_{a,b}$. Then it is not hard to see that

$$S_{a,b} = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I_{u_0}} c_i x_{i+k} \oplus \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\partial] \sum_{i \in I_{u_0}} c_i \delta_{i+k}(\partial) y_{i+k}.$$

□

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